arXiv:hep-th/0411053v2 5 May 2006

CLIFFORD SPACE AS A GENERALIZATION OF SPACETIME: PROSPECTS FOR UNIFICATION IN PHYSICS[1]

Matej Pavšič

Jožef Stefan Institute, Jamova 39, 1000 Ljubljana, Slovenia

e-mail: matej.pavsic@ijs.si

Abstract

The geometric calculus based on Clifford algebra is a very useful tool for geometry and physics. It describes a geometric structure which is much richer than the ordinary geometry of spacetime. A Clifford manifold (C-space) consists not only of points, but also of 1-loops, 2-loops, etc.. They are associated with multivectors which are the wedge product of the basis vectors, the generators of Clifford algebra. We assume that C-space is the true space in which physics takes place and that physical quantities are Clifford algebra valued objects, namely, superpositions of multivectors, called Clifford aggregates or polyvectors. We explore some very promising features of physics in Clifford space, in particular those related to a consistent construction of string theory and quantum field theory.

1. Introduction

In recent years there has been the interest in considering a theory in which spacetime is replaced by Clifford space [1]–[7]. The latter space describes not only points, but also areas, volumes, etc., altogether on the same footing. In refs. [5] it was proposed that the coordinates of Clifford space can be interpreted as generalizing the concept of center of mass coordinates of extended objects. In other words, the extended objects can be modeled by coordinates of Clifford space.

Instead of the usual theory of relativity in spacetime M_n we have the relativity in C-space. The latter space has dimension 2^n and signature $(+ + +... - - - ...)$, where the number of plus and minus signs is the same, namely $2^n/2$. This has consequences for string theory which can be formulated without central terms in Virasoro algebra even when the dimension of the underlying spacetime is four, provided that the Jackiw definition [8] of vacuum is employed [9]. We do not need a higher dimensional target spacetime for a

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consistent formulation of (quantized) string theory. Instead of a higher dimensional space we have Clifford space which also provides a natural framework [3, 4] for description of superstrings and supersymmetry, since spinors are just the elements of left or right minimal ideals of Clifford algebra [10]–[13].

When considering field theory and assuming the Jackiw definition of vacuum state, the concept of C-space enables a formulation in which zero point energies belonging to positive and negative signature degrees of freedom cancel out [9], while preserving the Casimir effect. This provides a resolution of the cosmological constant problem.

Instead of flat C-space we may consider a curved C-space. As the passage from flat Minkowski spacetime to curved spacetime had provided us with a tremendous insight into the nature of one of the fundamental interactions, namely gravity, so the introduction of a curved C-space will presumably even further increase our understanding of the other fundamental interactions and their unification with gravity.

2. On the Relativity in Clifford space

Let us start with n-dimensional flat spacetime M_n whose signature is $(+ - - - -...)$. A point P in M_n can be associated with a *vector* x joining the coordinate origin O and P. Vectors can be represented by Clifford numbers according to

$$x = x^\mu \gamma_\mu \tag{1}$$

where γ_μ satisfy the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu} \tag{2}$$

and $g_{\mu\nu}$ is the metric of M_n.

The latter relation is the *inner product* between two basis vectors γ_μ and γ_ν. It is just the *symmetric part* of the Clifford product $\gamma_\mu \gamma_\nu$.

Besides vectors we can consider *bivectors* defined by the wedge product $\gamma_\mu \wedge \gamma_\nu$ which is the *antisymmetric part* of $\gamma_\mu \wedge \gamma_\nu$:

$$\gamma_\mu \wedge \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{3}$$

and in general multivectors or r-vectors

$$\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r} = \frac{1}{r!}[\gamma_{\mu_1}, \gamma_{\mu_2}, ..., \gamma_{\mu_r}] \tag{4}$$

Bivectors, trivectors, etc., represent oriented areas, volumes, etc., respectively. The highest grade r in M_n is $r = n$. The r-vectors of $r > n$ are identically zero.

A generic *Clifford number* (called also *polyvector* of *Clifford aggregate*) is

$$X = \sigma \underline{1} + x^\mu \gamma_\mu + \frac{1}{2} x^{\mu_1 \mu_2} \gamma_{\mu_1 \mu_1} + ... + \frac{1}{n!} x^{\mu_1...\mu_n} \gamma_{\mu_1...\mu_n} \equiv x^M \gamma_M \tag{5}$$

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where $\gamma_{\mu_1...\mu_r} \equiv \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ and

$$x^M = (\sigma, x^{\mu_1}, x^{\mu_1 \mu_2}, ..., x^{\mu_1 \cdots \mu_r})$$
$$\gamma_M = (\underline{1}, \gamma_\mu, \gamma_{\mu_1 \mu_2}, ..., \gamma_{\mu_1 ... \mu_r}), \quad \mu_1 < \mu_2 < ... < \mu_r \tag{6}$$

are respectively coordinates and basis elements of Clifford algebra.

The coordinates $x^{\mu_1 \cdots \mu_r}$ determine an oriented r-area. They say nothing about the precise form of the $(r-1)$-loop enclosing the r-area. The coordinates σ, x^{μ_1}, $x^{\mu_1 \mu_2}$, ... provide a means for a description of extended objects. If an objects is extended, then not only its center of mass coordinates x^μ, but also the higher grade coordinates $x^{\mu_1 \mu_2}$, $x^{\mu_1 \mu_2, \mu_3}$,..., associated with the object extension, are different from zero, in general. Those higher grade coordinates model the extended object. A detailed description is provided in ref. [5].

Since x^M assume any real value, the set of all possible X forms a 2^n-dimensional manifold, called *Clifford space*, or shortly *C-space*.

Let us define the quadratic form by means of the scalar product

$$|\mathrm{d}X|^2 \equiv \mathrm{d}X^\ddagger * \mathrm{d}X = \mathrm{d}x^M \mathrm{d}x^N G_{MN} \equiv \mathrm{d}x^M \mathrm{d}x_M \tag{7}$$

where the metric of C-space is given by

$$G_{MN} = \gamma_M^\ddagger * \gamma_N \tag{8}$$

The operation $\ddagger$ reverses the order of vectors:

$$(\gamma_{\mu_1} \gamma_{\mu_2} ... \gamma_{\mu_r})^\ddagger = \gamma_{\mu_r} ... \gamma_{\mu_2} \gamma_{\mu_1} \tag{9}$$

Indices are lowered and raised by G_{MN} and its inverse G^{MN}, respectively. The following relation is satisfied:

$$G^{MJ} G_{JN} = \delta^M{}_N \tag{10}$$

Eq.(7) is the expression for the *line element* in C-space. If C-space is generated from the basis vectors γ_μ of spacetime M_n with signature $(+ - - - - - ...)$, then the signature of C-space is $(+ + + ... - - - ...)$, where the nuber of plus and minus signs is the same, namely, $2^n/2$. This has some important consequences that we are going to investigate.

We assume that 2^n-dimensional Clifford space is the arena in which physics takes place. We can take $n = 4$, so that the spacetime from which we start is just the 4-dimensional Minkowski space M_4. The corresponding Clifford space has then 16 dimensions. In C-space the usual points, lines, surfaces, volumes and 4-volumes are all described on the same footing and can be transformed into each other by rotations in C-space (called polydimensional rotations):

$$x'^M = L^M{}_N x^N \tag{11}$$

subjected to the condition $|\mathrm{d}X'|^2 = |\mathrm{d}X|^2$.

We can now envisage that physical objects are described in terms of $x^M = (\sigma, x^\mu, x^{\mu\nu}, ...)$. The first straightforward possibility is to introduce a single parameter τ and consider a mapping

$$\tau \to x^M = X^M(\tau) \tag{12}$$

where $X^M(\tau)$ are 16 embedding functions that describe a worldline in C-space. From the point of view of C-space, $X^M(\tau)$ describe a wordlline of a "point particle": At every value of τ we have a *point* in C-space. But from the perspective of the underlying 4-dimensional spacetime, $X^M(\tau)$ describe an extended object, sampled by the center of mass coordinates $X^\mu(\tau)$ and the coordinates $X^{\mu_1\mu_2}(\tau), ..., X^{\mu_1\mu_2\mu_3\mu_4}(\tau)$. They are a generalization of the center of mass coordinates in the sense that they provide information about the object 2-vector, 3-vector, and 4-vector extension and orientation[2].

The dynamics of such an object is determined by the action

$$I[X] = M \int \mathrm{d}\tau \, (\dot{X}^\ddagger * \dot{X})^{\frac{1}{2}} = M \int \mathrm{d}\tau (\dot{X}^M \dot{X}_M)^{\frac{1}{2}} \tag{13}$$

The dynamical variables are given by the polyvector

$$X = X^M \gamma_M = \sigma\underline{1} + X^\mu \gamma_\mu + X^{\mu_1\mu_2}\gamma_{\mu_1\mu_2} + ... X^{\mu_1...\mu_n}\gamma_{\mu_1...\mu_n} \tag{14}$$

whilst

$$\dot{X} = \dot{X}^M \gamma_M = \dot{\sigma}\underline{1} + \dot{X}^\mu \gamma_\mu + \dot{X}^{\mu_1\mu_2}\gamma_{\mu_1\mu_2} + ... \dot{X}^{\mu_1...\mu_n}\gamma_{\mu_1...\mu_n} \tag{15}$$

is the velocity polyvector, where $\dot{X}^M \equiv \mathrm{d}X^M/\mathrm{d}\tau$.

In the action (13) we have a straightforward generalization of the relativistic point particle in M_4:

$$I[X^\mu] = m \int \mathrm{d}\tau (\dot{X}^\mu \dot{X}_\mu)^{\frac{1}{2}} \, , \quad \mu = 0, 1, 2, 3 \tag{16}$$

If a particle is extended, then (16) provides only a very incomplete description. A more complete description is given by the action (13), in which the C-space embedding functions $X^M(\tau)$ sample the objects extension.

3. Strings and Clifford space

Usual strings are described by the mapping $(\tau, \sigma) \to x^\mu = X^\mu(\tau, \sigma)$, where the embedding functions $X^\mu(\tau, \sigma)$ describe a 2-dimensional worldsheet swept by a string. The action is given by the requirement that the area of the worldsheet be "minimal" (extremal). Such action is invariant under reparametrizations of (τ, σ). There are several equivalent forms of the action including the "σ-model action" which, in the conformal gauge, can be written as

$$I[X^\mu] = \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\dot{X}^\mu \dot{X}_\mu - X'^\mu X'_\mu) \tag{17}$$

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[2]A systematic and detailed treatment is in ref. [5].

where $\dot{X}^\mu \equiv dX^\mu/d\tau$ and $X'^\mu \equiv dX^\mu/d\sigma$. Here κ is the string tension, usually written as $\kappa = 1/(2\pi\alpha')$.

String coordinates X^μ and momenta $P_\mu = \partial L/\partial\dot{X}^\mu = \kappa\dot{X}_\mu$ satisfy the following constraints ($\sigma \in [0, \pi]$):

$$\varphi_1(\sigma) = P^\mu P_\mu + \frac{X'^\mu X'_\mu}{(2\pi\alpha')^2} \approx 0 \qquad \varphi_2(\sigma) = \frac{P^\mu X'_\mu}{\pi\alpha'} \approx 0 \qquad (18)$$

which can be written as a single constraint on the interval $\sigma \in [-\pi, \pi]$

$$\Pi^\mu\Pi_\mu(\sigma) \approx 0 \qquad \Pi^\mu = P^\mu + \frac{X'^\mu}{2\pi\alpha'} \qquad (19)$$

to which the open string is symmetrically extended. (For more details see the literature on strings, e.g., [14].)

If we generalize the action (17) to C-space, we obtain

$$I[X] = \frac{\kappa}{2}\int d\tau\, d\sigma\, (\dot{X}^M\dot{X}^N - X'^M X'_N)G_{MN} \qquad (20)$$

where κ is the generalized string tension. Taking 4-dimensional spacetime, there are $D = 2^4 = 16$ dimensions of C-space. Its signature $(+++... ----...)$ has 8 plus and 8 minus signs.

Let us consider the case of open string satisfying the boundary condition $X'^M = 0$ at $\sigma = 0$ and $\sigma = \pi$. Then we can make the expansion

$$X^M(\tau, \sigma) = \sum_{n=-\infty}^{\infty} X_n^M(\tau)\, e^{in\sigma} \qquad (21)$$

where from the reality condition $(X^M)^* = X^M$ it follows

$$X_n^M = X_{-n}^M \qquad (22)$$

Inserting (21) into (20), integrating over σ and taking into account (22) we obtain the action expressed in terms of $X_n^M(\tau)$:

$$I[X_n^M] = \frac{\kappa'}{2}\int d\tau\, \sum_{n=-\infty}^{\infty} (\dot{X}_n^M\dot{X}_n^N - n^2 X_n^M X_n^N)G_{MN} \qquad (23)$$

where $\kappa' = 2\pi\kappa = 1/\alpha'$. This is just the action of infinite number of harmonic oscillators.

The Hamiltonian corresponding to the action (23) is

$$H = \frac{1}{2}\sum_{n=-\infty}^{\infty} \left(\frac{1}{\kappa'}P_n^M P_{nM} + \kappa' n^2 X_n^M X_{nM}\right) \qquad (24)$$

Let us introduce

$$a_n^M = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\kappa'}}P_n^M - in\sqrt{\kappa'}\, X_n\right)$$

$$a_n^{M\dagger} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{\kappa'}}P_n^M + in\sqrt{\kappa'}\, X_n\right) \qquad (25)$$

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We see that $a_{-n}^M = a_N^{M\dagger}$. Rewriting H in terms of a_n^M, $a_n^{M\dagger}$ we obtain

$$H = \frac{1}{2} \sum_{n=-\infty}^{\infty} (a_n^{M\dagger} a_{nM} + a_{nM} a_n^{M\dagger}) = \sum_{n=1}^{\infty} (a_n^{M\dagger} a_{nM} + a_{nM} a_n^{M\dagger}) + \frac{1}{2\kappa'} P_0^M P_{0M} \tag{26}$$

Upon quantization we have

$$[X_n^M, P_{nN}] = i\delta^M{}_N \quad \text{or} \quad [X_n^M, P_n^N] = iG^{MN} \tag{27}$$

and

$$[a_n^M, a_{nN}^{\dagger}] = i\delta^M{}_N \quad \text{or} \quad [a_n^M, a_n^{N\dagger}] = G^{MN} \tag{28}$$

In order to construct the Fock space of excited states, one has first to define a *vacuum state*. There are two possible choices [9].

Possibility I. Conventionally, vacuum state is defined according to

$$a_n^M |0\rangle = 0 , \qquad n \geq 1 \tag{29}$$

and the excited part of the Hamiltonian $H_{\text{exc}} = H - (1/\kappa') P_0^M P_{0M}$, after using (28) and (29) is

$$H_{\text{exc}} = \sum_{n=-\infty}^{\infty} (a_n^{M\dagger} a_{nM} + \frac{D}{2}) = 2 \sum_{n=1}^{\infty} (a_n^{M\dagger} a_{nM} + \frac{D}{2}) \tag{30}$$

$$D = \delta^M{}_M = G^{MN} G_{MN}$$

Its eigenvalues are all positive[3] and there is the non vanishing zero point energy. But there exist negative norm states.

Possibility II. Let us split $a_n^M = (a_n^A, a_n^{\bar{A}})$ where the indices A, $\bar{A}$ refer to the components with positive and negative signature, respectively, and let us define vacuum according to

$$a_n^A |0\rangle = 0 , \qquad (a_n^{\bar{A}})^{\dagger} |0\rangle = 0 , \quad n \geq 1 \tag{31}$$

Using (28) we obtain the following Hamlitonian, in which annihilation operators, defined in eq. (31), are on the right:

$$H_{\text{exc}} = 2 \sum_{n=1}^{\infty} (a_n^{A\dagger} a_{nA} + \frac{R}{2} + a_{n\bar{A}} a_n^{\bar{A}\dagger} - \frac{S}{2}) \tag{32}$$

where $R = \delta_A{}^A$ and $S = \delta_{\bar{A}}{}^{\bar{A}}$. *There are no negative norm states.*

If the number of positive and negative signature components is the same, i.e., $R = S$, then the above Hamiltonian has vanishing zero point energy:

$$H_{\text{exc}} = 2 \sum_{n=1}^{\infty} (a_n^{A\dagger} a_{nA} + a_{n\bar{A}} a_n^{\bar{A}\dagger}) \tag{33}$$

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[3]This is so even for those components a_n^M that belong to negative signature: negative sign of a term in $a_n^{M\dagger} a_{nM}$ is compensated by negative sign in the commutation relation (28).

Its eigenvalues can be positive or negative, depending on which components (positive or negative signature) are excited.

An immediate objection could arise at this point, namely, that since the spectrum of the Hamiltonian is not bounded from below, the system described by H of eq.(32) or (33) is unstable. This objection would only hold if the kinetic terms $\dot{X}_n^M \dot{X}_{nM}$ in the action (23) (or the terms $P_n^M p_{nM}$ in the Hamiltonian (24)) were all positive, so that negative eigenvalues of H would come from the negative potential terms in $n^2 X_n^M X_{nM}$. But since our metric is pseudo-Euclidean, whenever a term in the potential is negative, also the corresponding kinetic term is negative. Therefore, the acceleration corresponding to negative signature term is proportional to the *plus* gradient of potential (and not to the minus gradient of potential as it is the case for positive signature term); such system is *stable* if potential has *maximum*, i.e., if it has an upper bound (and not a lower bound). The overall change of sign of the action (Lagrangian) has no influence on the equations of motion (and thus on stability).

In the bosonic string theory based on the ordinary definition of vacuum (Possibility I) and formulated in D-dimensional spacetime with signature $(+ - - - ... - --)$ there are negative norm states, unless $D = 26$. Consistency of the string theory requires extra dimensions, besides the usual four dimensions of spacetime.

My proposal is that, instead of adding extra dimensions to spacetime, we can start from 4-dimensional spacetime M_4 with signature $(+ - --)$ and consider the Clifford space $\mathcal{C}_{M_4}$ (C-space) whose dimension is 16 and signature $(8+, 8-)$. *The necessary extra dimensions for consistency of string theory are in C-space.* This also automatically brings *spinors* into the game. It is an old observation that spinors are the elements of left or right ideals of Clifford algebras [10]–[12] (see also a very lucid and systematic recent exposition in refs. [13]). In other words, spinors are particular sort of polyvectors [4]. Therefore, the string coordinate polyvectors contain spinors. This is an alternative way of introducing spinors into the string theory [4, 15].

Let the constraints (18),(19) be generalized to C-space. So we obtain

$$\Pi^M \Pi_M \approx 0 \,, \qquad \Pi^M = P^M + \frac{X'^M}{2\pi\alpha'} \tag{34}$$

Using (21) and expanding the momentum $P^M(\sigma)$ according to

$$P^M = \sum_{n=-\infty}^{\infty} P_n^M e^{in\sigma} \tag{35}$$

we can calculate the Fourier coefficients of the constraint (called Virasoro generators):

$$L_n = \frac{\pi\alpha'}{2} \int_{-\pi}^{\pi} d\sigma \, e^{-in\sigma} \, \Pi^M \Pi_M = \frac{1}{2} \sum_{r=-\infty}^{\infty} a_n^M a_{r-n}^N G_{MN} \tag{36}$$

If we calculate the commutators of Virasoro generators, then, after putting to the left those operators which according to the vacuum definition (31) act

as creation operators, we obtain the following relation

$$[L_m, L_n] = (m - n)L_{n+m} \tag{37}$$

in which there are no central terms. The terms which arise after reordering of operators have opposite signs for positive and negative signature components, and thus cancel out. The algebra of Virasoro generators is thus closed, which automatically assures consistency of quantum string theory.

4. Quantum Field Theory and C-space

In previous section we considered string theory which can be considered as a theory of D fields $X^M(\tau, \sigma)$ over a 2-dimensional space. Let us now turn to the theory of n scalar fields $\phi^a(x^\mu)$, $a = 0, 1, 2, ..., n - 1$, over the 4-dimensional spacetime parametrized by coordinates x^μ, $\mu = 0, 1, 2, 3$. The action for such system is

$$I[\phi^a] = \frac{1}{2} \int d^4x \sqrt{-g}(g^{\mu\nu}\partial_\mu\phi^a \partial_\nu\phi^b - m^2\phi^a\phi^b)\gamma_{ab} \tag{38}$$

where γ_{ab} is the metric in the space of fields ϕ^a, and $g_{\mu\nu}$ the metric of spacetime. Let us assume that $g_{\mu\nu} = \eta_{\mu\nu}$ is the metric of flat spacetime

The canonical momenta are

$$\pi_a = \frac{\partial\mathcal{L}}{\partial\partial_0\phi^a} = \partial^0\phi_a = \partial_0\phi_a \equiv \dot{\phi}_a \tag{39}$$

Upon quantization the following equal time commutation relations are satisfied:

$$[\phi^a(\mathbf{x}), \pi_b(\mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}')\delta^a{}_b \tag{40}$$

The Hamiltonian is given by

$$H = \frac{1}{2} \int d^3x \, (\dot{\phi}^a\dot{\phi}^b - \partial_i\phi^a\partial^i\phi^b + m^2\phi^a\phi^b)\gamma_{ab} \tag{41}$$

where $i = 1, 2, ..., n - 1$. We shall assume that γ_{ab} is diagonal. A general solution to the equations of motion derived from the action (38) can be written in the form

$$\phi^a = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{k}}(a^a(\mathbf{k})e^{-ikx} + a^{a\dagger}(\mathbf{k})e^{ikx}) \tag{42}$$

Here[4] $\omega_\mathbf{k} \equiv |\sqrt{m^2 + \mathbf{k}^2}|$. The creation and annihilation operators satisfy the commutation relations

$$[a^a(\mathbf{k}), a_b{}^\dagger(\mathbf{k})] = (2\pi)^3 2\omega_\mathbf{k} \delta^3(\mathbf{k} - \mathbf{k}')\delta^a{}_b \tag{43}$$

or

$$[a^a(\mathbf{k}), a^{b\dagger}(\mathbf{k})] = (2\pi)^3 2\omega_\mathbf{k} \delta^3(\mathbf{k} - \mathbf{k}')\gamma^{ab} \tag{44}$$

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[4]We use units in which $\hbar = c = 1$.

Inserting the expansion (42) of fields ϕ^a into the Hamiltonian (41) we obtain

$$H = \frac{1}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega_\mathbf{k}}{2\omega_\mathbf{k}} (a^{a\dagger}(\mathbf{k})a^b(\mathbf{k}) + a^a(\mathbf{k})a^{b\dagger}(\mathbf{k}))\gamma_{ab} \qquad (45)$$

Let us assume that the signature of the metric γ_{ab} is pseudo-Euclidean, and let us write

$$a^a(\mathbf{k}) = (a^\alpha, a^{\bar{\alpha}}) \qquad (46)$$

where α denotes positive and $\bar{\alpha}$ negative signature components.

We will define vacuum according to

$$a^\alpha(\mathbf{k})|0\rangle = 0 \ , \quad a^{\bar{\alpha}\dagger}(\mathbf{k})|0\rangle = 0 \qquad (47)$$

If we reorder the operators in the Hamiltonian (45) so that the annihilation operators (with respect to the vacuum definition (47)) are on the right, we find

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega_\mathbf{k}}{2\omega_\mathbf{k}} (a^{\alpha\dagger}(\mathbf{k})a_\alpha(\mathbf{k}) + a^{\bar{\alpha}}(\mathbf{k})a_{\bar{\alpha}}{}^\dagger(\mathbf{k}) + \frac{r}{2} - \frac{s}{2}) \qquad (48)$$

where $r = \delta^\alpha{}_\alpha$ and $s = \delta^{\bar{\alpha}}{}_{\bar{\alpha}}$. In the case in which the signature has equal number of plus and minus signs, i.e., when $r = s$, the zero point energies cancel out from the Hamiltonian.

Now a question arises as to why should the space of fields have the metric with $r = s$. Isn't it an ad hoc assumption? As in the case of string we can consider the space of fields V_n just as a starting space, with basis e_a, $a = 0, 1, 2, ..., n - 1$, from which we generate the 2^n-dimensional Clifford space $\mathcal{C}_{V_n}$ with basis $e_A = (\underline{1}, e_a, e_{a_1 a_2}, ..., e_{a_1...a_n})$, $a_1 < a_2 < ... < a_n$. If V_n is a Euclidean space so that $e_a \cdot e_b = \delta_{ab}$ is the Euclidean metric, then also the metric $e_A{}^\ddagger * e_B$ of $\mathcal{C}_{V_n}$ is Eucliddean. But, as it was pointed out in refs. [3, 4], instead of the basis e_A we can take another basis, e.g.,

$$\gamma_A = (\underline{1}, \gamma_a, \gamma_{a_1 a_2}, ..., \gamma_{a1...a_n}) \qquad (49)$$

generated from the set of Clifford numbers $\gamma_a = (e_0, e_i e_0)$, $a = 0, 1, 2, ..., n - 1$; $i = 1, 2, ..., n$ satisfying

$$\gamma_a \cdot \gamma_b \equiv \frac{1}{2}(\gamma_a \gamma_b + \gamma_b \gamma_a) = \eta_{ab} \qquad (50)$$

where

$$\gamma_{a_1 a_2...a_r} \equiv \gamma_{a_1} \wedge \gamma_{a2} \wedge ... \wedge \gamma_{a_r} \equiv \frac{1}{2}[\gamma_{a1}, \gamma_{a2}, ..., \gamma_{a_r}] \qquad (51)$$

The metric

$$\gamma_A{}^\ddagger * \gamma_B = G_{AB} \qquad (52)$$

defined with respect to the new basis is pseudo-Euclidean, its signature having $2^n/2$ plus and $2^n/2$ minus signs. For simplicity we use in eq.(52) the same symbol '$\ddagger$', but now it denotes reversion of new basis vectors: $(\gamma_{a_1}...\gamma_{a_r})^\ddagger = \gamma_{a_r}...\gamma_{a_1}$.

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We assume that a field theory should be formulated in C-space. Instead of the action (38) we thus consider its generalization to C-space:

$$I = \frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^A \partial_\nu \phi^B - m^2 \phi^A \phi^B) G_{AB} \qquad (53)$$

Here $\Phi = \phi^A \gamma_A$ is a polyvector field and $\Phi^\ddagger * \Phi = \phi^A \phi^B G_{AB}$. Since the metric G_{AB} has signature $(+++...---...) = (R+, S-)$ with $R = S$, zero point energies of a system based on the action (53) cancel out: Vacuum energy vanishes. Consequently, in such a theory there is no cosmological problem [9]. The small cosmological constant, as recently observed, could be a residual effect of something else.

Cancellation of vacuum energies in the theory does not exclude [9] the existence of well known effects, such as Casimir effect, which are manifestation of vacuum energies.

5. Discussion and Conclusion

We started from 4-dimensionl spacetime M_4 and generalized it to Clifford space $\mathcal{C}_{M_4}$. We considered a theory of a 1-dimensional worldline (swept by point particle) and of a 2-dimensional worldsheet (swept by string) living in 16-dimensional Clifford space. In this theory no extra dimensions of the target spacetime are required. The necessary extra dimensions are in Clifford space (C-space) generated by 4 independent basis vectors γ_μ of spacetime. So we obtain a framework in which fermions (as elements of the minimal ideals of $\mathcal{C}_{M_4}$) also enter the game. A next logical step is to generalize the $(p+1)$-dimensional world manifold (including, for $p = 1$, the string worldsheet V_2) to to the corresponding Clifford space $\mathcal{C}_{V_{p+1}}$. Such theory is discussed in ref. [4].

We then consider the quantum theory of n scalar fields and take the space of fields as a starting space from which we construct the corresponding Clifford space $\mathcal{C}_{V_n}$. Amongst available 2^n basis elements of $\mathcal{C}_{V_n}$ we are free to choose n elements, denoted γ_a, $a = 0, 1, 2, ..., n$, such that the inner products defined according to eq.(50) form the Minkowski metric with signature $(+---...)$. Taking γ_a as generators of Clifford algebra and define the metric of C-space according to eq. (52), we obtain that the signature of Clifford space is $(+++...---...)$ with 2^n plus and 2^n minus signs. We then find that the vacuum energy belonging to the negative signature degrees of freedom cancel out the vacuum energy belonging to the positive signature degrees of freedom. In the usual quantum field theory vacuum energy is infinite (or given by the Planck scale cutoff). Since any form of energy is coupled to gravity, as a result we obtain the cosmological constant which is drastically too high (120 orders of magnitude) in comparison with the experimentally observed value. In our theory with vanishing vacuum energy, there is no cosmological constant problem. In ref.[9] it is shown why such cancellation of vacuum energies is not in disagreement with the Casimir effect and other effects of vacuum. In our field theory fermions are automatically present as the elements of the minimal ideals of $\mathcal{C}_{V_n}$. In the usual theory fermions are included in order to satisfy the requirement of

supersymmetry. In supersymmetric field theory similar cancellation of vacuum energies occurs.

In the quantum field theory discussed here only the space of field V_n has been generalized to Clifford space. A next step is to generalize spacetime, in which those fields live, as well, and consider a full quantum field theory in C-space. A generalization to string field theory also remains to be investigated.

In ref.[15] curved C-space was considered and it was found out that the Einstein-Hilbert action, generalized to C-space, contained the ordinary higher derivative gravity in 4-dimensional spacetime. Curved C-space also appears very promising for the unification of fundamental interactions because it automatically provides the required extra dimensions which in Kaluza-Klein theories are postulated ad hoc and added to the known four dimensions of spacetime. Following our approach, we can remain with four dimensions of spacetime. Once we have a 4-dimensional spacetime, the 16-dimensional Clifford space is automatically there: It is constructed from spacetime by means of its basis vectors γ_μ which are generators of Clifford algebra. We only need to employ Clifford space for formulation of physical theories. A part of this project is described in this paper, more in refs. [1]–[7],[15], and much more remains to be done.

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